THE FAILURE OF THE CLIFFORD CHAIN.

By Walter B. Carver.

The Clifford chain theorem* defines, for a set of n lines in a plane no two of which are parallel, a Clifford circle when n is odd and a Clifford point when n is even. The Clifford point for two lines is their point of intersection, and the Clifford circle for three lines is the circum-circle. For any odd n, each set of $n - 1$ lines out of the n lines determines a Clifford point, and the n such Clifford points lie on a circle, the Clifford circle of the n lines. For any even n (greater than 2), each set of $n - 1$ lines determines a Clifford circle, and the n such Clifford circles pass through a point, the Clifford point of the n lines. In his proof of the theorem, Clifford does not raise the question of the existence of exceptional sets of lines for which the theorem may fail.

Kantor pointed out† that when five lines touch a deltoid (a hypocycloid of three cusps) the five Clifford points of the sets of four lines lie on a straight line instead of a circle; and that for six lines touching a deltoid, the six Clifford *lines* for the sets of five lines are not concurrent but are tangents of a new deltoid.

In his paper "On the Metric Geometry of the Plane N-Line,"‡ Morley gives, incidentally, an analytic proof of Clifford's theorem; shows that any odd number of lines (greater than three) determine, when they satisfy a certain analytic condition, a Clifford *line* rather than a circle; and intimates that further degeneracy may occur.

It is the purpose of the present paper to consider all the possibilities of failure of the Clifford chain, and to examine the conditions on a set of lines which cause such failure.

§ 1. *Characteristic Constants; Map and Envelope Equations.*

The analysis used will be the circular coördinates and characteristic constants of Morley's paper. It will be necessary to give a few preliminary definitions, and to state certain fundamental relations to be used later. These relations will be stated without proof, as some of them may be found

* Clifford, "Synthetic Proof of Miquel's Theorem," *Messenger of Mathematics*, Vol. 5, page 124; 1870.

† Kantor, "Die Tangentengeometrie an der Steiner'schen Hypocycloid," *Wiener Sitzungsberichte*, Vol. 78, page 204; 1878.

‡ *Transactions of the Amer. Math. Soc.*, Vol. 1, page 97; 1900.

in Morley's paper and the others may be verified by the reader without difficulty.

The circular coördinates x and y of a point are defined by the equations $x = X + iY$, $y = X - iY$, where X, Y are rectangular Cartesian coördinates and i is the imaginary unit. A complex number whose absolute value is unity will be called a turn. A real line is represented by a linear equation $tx + y = ct$, where c is any complex number and t a turn such that $\bar{c} = ct$. The point* c is the reflexion of the origin in the line (briefly, the reflex point of the line), and the turn t gives the inclination of the line and will be called the clinant. Two lines with clinants t_1 and t_2 are parallel if $t_1 = t_2$, and perpendicular if $t_1 = -t_2$.

A set of n lines (no two of which are parallel) determine uniquely a set of n characteristic constants, defined as follows:

$$a_i = \frac{1}{T}\begin{vmatrix} c_1t_1^{n-i} & t_1^{n-2} & t_1^{n-3} & \cdots & t_1 & 1 \\ c_2t_2^{n-i} & t_2^{n-2} & t_2^{n-3} & \cdots & t_2 & 1 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ c_nt_n^{n-i} & t_n^{n-2} & t_n^{n-3} & \cdots & t_n & 1 \end{vmatrix}, \qquad i = 1, 2, \cdots, n,$$

where

$$T = \begin{vmatrix} t_1^{n-1} & t_1^{n-2} & \cdots & t_1 & 1 \\ t_2^{n-1} & t_2^{n-2} & \cdots & t_2 & 1 \\ \cdots & & & & \\ t_n^{n-1} & t_n^{n-2} & \cdots & t_n & 1 \end{vmatrix}.$$

Between these n a's we have the relations

$$\bar{a}_i = (-1)^{n-1}S_n a_{n-i+1}, \qquad i = 1, 2, \cdots, n,$$

where $S_n = t_1t_2t_3\cdots t_n$. Since $S_n \neq 0$, a_i and a_{n-i+1} (called complementary a's) vanish together.

To fix an arbitrary set of n lines,

(1) for n even, $a_1, a_2, \cdots, a_{n/2}$ and the n clinants $t_1, t_2, \cdots, t_n$ (no two of them equal) may be chosen arbitrarily; and the remaining a's and the set of n lines are then uniquely fixed:

(2) for n odd, $a_1, a_2, \cdots, a_{(n+1)/2}$, and any $n-1$ of the n clinants $t_1, t_2, \cdots, t_n$ may be chosen arbitrarily; and the remaining t and a's and the set of n lines are then uniquely fixed.

If one of a set of n lines is omitted, the constants for the remaining set of $n - 1$ lines are

$$\alpha_i = a_i - ta_{i+1}, \qquad i = 1, 2, \cdots, n-1,$$

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* The phrase "the point c" is used in the sense of the usual relation between the complex numbers and the points of a plane. The point c is the *real* point whose circular coördinates are $(c, \bar{c})$.

where t is the clinant of the omitted line. More generally, if r lines are omitted, the constants for the $n - r$ lines are

$$\alpha_i = a_i - S_1 a_{i+1} + S_2 a_{i+2} - \cdots + (-1)^r S_r a_{i+r}, \qquad i = 1, 2, \cdots, n - r,$$

where the S's are the ordinary symmetric functions* of the r clinants of the omitted lines.

A set of n lines are concurrent if, and only if,

$$a_2 = a_3 = \cdots = a_{n-1} = 0;$$

and the lines will then meet at the point a_1. If $n - r$ lines out of a set of n lines are concurrent, we must then have

$$\begin{aligned}
a_2 - S_1 a_3 + S_2 a_4 - \cdots + (-1)^r S_r a_{r+2} &= 0,\\
a_3 - S_1 a_4 + S_2 a_5 - \cdots + (-1)^r S_r a_{r+3} &= 0,\\
&\vdots\\
a_{n-r-1} - S_1 a_{n-r} \quad - \cdots + (-1)^r S_r a_{n-1} &= 0,
\end{aligned}$$

and the $n - r$ lines meet at the point

$$a_1 - S_1 a_2 + \cdots + (-1)^r S_r a_{r+1};$$

where the S's are for the clinants of the omitted lines. If $n - r \leqq (n/2) + 1$, this imposes no condition on the a's alone; but if $n - r > (n/2) + 1$, the matrix

$$\left\| \begin{matrix} a_2 & a_3 & \cdots & a_{r+2} \\ a_3 & a_4 & \cdots & a_{r+3} \\ \cdot & \cdot & \cdot & \cdot \\ a_{n-r-1} & a_{n-r} & \cdots & a_{n-1} \end{matrix} \right\|$$

must be of rank r if $n - r$ (and no more) of the n lines are concurrent.

In the following sections an important rôle will be played by determinants of the form

$$\begin{vmatrix} a_i & a_{i+1} & \cdots & a_{i+k-1} \\ a_{i+1} & a_{i+2} & \cdots & a_{i+k} \\ \cdot & \cdot & \cdot & \cdot \\ a_{i+k-1} & a_{i+k} & \cdots & a_{i+2k-2} \end{vmatrix}$$

where the elements are the characteristic constants of a set of n lines. Such a determinant of the kth order is equal to a determinant of the nth order in the reflex points and clinants of the n lines, as follows:

$$\begin{vmatrix} a_i & a_{i+1} & \cdots & a_{i+k-1} \\ a_{i+1} & a_{i+2} & \cdots & a_{i+k} \\ \cdot & \cdot & \cdot & \cdot \\ a_{i+k-1} & a_{i+k} & \cdots & a_{i+2k-2} \end{vmatrix}$$

* Throughout the paper S_i, $i = 1, 2, \cdots k$, will represent the ordinary symmetric functions of k turns $t_1, t_2, \cdots t_k$; *i.e.*, the S's will be the coefficients of the equation $t^k - S_1 t^{k-1} + S_2 t^{k-2} - S_3 t^{k-3} + \cdots + (-1)^k S_k = 0$ whose roots are the k turns.

$$= \frac{(-1)^{k(k-1)/2}}{T} \begin{vmatrix} c_1 t_1^{n-i-k+1} & c_1 t_1^{n-i-k} & \cdots & c_1 t_1^{n-i-2k+2} & t_1^{n-k-1} & t_1^{n-k-2} & \cdots & t_1 & 1 \\ c_2 t_2^{n-i-k+1} & c_2 t_2^{n-i-k} & \cdots & c_2 t_2^{n-i-2k+2} & t_2^{n-k-1} & t_2^{n-k-2} & \cdots & t_2 & 1 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ c_n t_n^{n-i-k+1} & c_n t_n^{n-i-k} & \cdots & c_n t_n^{n-1-2k+2} & t_n^{n-k-1} & & \cdots & t_n & 1 \end{vmatrix}$$

where

$$T = \begin{vmatrix} t_1^{n-1} & t_1^{n-2} & \cdots & t_1 & 1 \\ t_2^{n-1} & t_2^{n-2} & \cdots & t_2 & 1 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ t_n^{n-1} & t_n^{n-2} & \cdots & t_n & 1 \end{vmatrix}.$$

Also such a determinant in the characteristic constants (α's) for $n-1$ out of n lines may be expressed in terms of the characteristic constants (a's) for the n lines thus

$$\begin{vmatrix} \alpha_i & \alpha_{i+1} & \cdots & \alpha_{i+k-1} \\ \alpha_{i+1} & \alpha_{i+2} & \cdots & \alpha_{i+k} \\ \cdot & \cdot & \cdot & \cdot \\ \alpha_{i+k-1} & \alpha_{i+k} & \cdots & \alpha_{i+2k-2} \end{vmatrix} = \begin{vmatrix} a_i & a_{i+1} & \cdots & a_{i+k-1} & a_{i+k} \\ a_{i+1} & a_{i+2} & \cdots & a_{i+k} & a_{i+k+1} \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ a_{i+k-1} & a_{i+k} & \cdots & a_{i+2k-2} & a_{i+2k-1} \\ t^k & t^{k-1} & \cdots & t & 1 \end{vmatrix}$$

where t is the clinant of the omitted line.

The determinants*

$$\begin{vmatrix} a_i & \cdots & a_{i+k-1} \\ \cdot & \cdot & \cdot \\ a_{i+k-1} & \cdots & a_{i+2k-2} \end{vmatrix} \quad \text{and} \quad \begin{vmatrix} a_{n-i-2k+3} & \cdots & a_{n-i-k+2} \\ \cdot & \cdot & \cdot \\ a_{n-i-k+2} & \cdots & a_{n-i+1} \end{vmatrix}$$

will be called complementary. Two such complementary determinants must obviously vanish together.

The rational curves which occur in the paper will be represented either by "map" equations or by "envelope" equations. The map equation

$$x = R(t)$$

expresses x as a rational function of a turn parameter t. This equation always implies the corresponding equation

$$y = \bar{R}(1/t),$$

where $\bar{R}$ means the function obtained by replacing the coefficients in the function R by their conjugates. When t runs through turn values, x gives the *real* points of the curve and x and y are conjugates. If t is given values other than turn values, x and y will not (in general) be conjugates, but will be the circular coördinates of *imaginary* points of the curve.

* Determinants of this type will be sufficiently indicated by giving only the four corner elements.

The following special map equations may be noted:

$$x = c + kt$$

is a circle with center at c and radius $|\, k \,|$.

$$x = c + \frac{k}{(1 + kt/\bar{k})^2}$$

is a parabola of which the point c is the focus and k represents a vector from the focus to its reflexion in the directrix. In this form the parameter t is the clinant of the tangent to the curve at the point x.

$$x = c + k\left(\frac{\bar{k}^2}{k^2t^2} - \frac{2kt}{\bar{k}}\right)$$

is a deltoid of which c is the center and $3k$ represents a vector from the center to one of the cusps. In this case also t is the clinant of the tangent at the point x.

The envelope equation of a curve is of the form

$$tx + y = R(t),$$

where R is a rational function satisfying the identical relation

$$\bar{R}(1/t) = R(t)/t.$$

For a given turn value of t the envelope equation represents a straight line, and as t runs through turn values this line envelopes the curve. The map equation of the curve is obtained at once from this envelope equation by partial differentiation with respect to t, giving

$$x = R'(t),$$

where the parameter t is the clinant of the tangent at the point x.

The envelope equation of a parabola is

$$tx + y = (\alpha t^2 + rt + \bar{\alpha})/(\gamma t + \bar{\gamma}),$$

where r is real, $\gamma \neq 0$, and the denominator is not a factor of the numerator. Similarly, the envelope equation of a deltoid is

$$tx + y = (\alpha_1 t^3 + \alpha_2 t^2 + \bar{\alpha}_2 t + \bar{\alpha}_1)/rt,$$

where r is real and $\neq 0$, and $\alpha_1 \neq 0$. Differentiating these equations with respect to t, and identifying the results with the map equations of these same curves, we see that the focus of the parabola is at the point α/γ; while the center of the deltoid is at α_2/r, and its size is given by $|\, \alpha_1 \,|/r$.

The most general curve which can be represented by an envelope equation is treated in sections 3 and 8.

§ 2. *Improper Sets of* 3, 4, 5, *and* 6 *Lines.*

The map equation of the Clifford circle as given by Morley is, for 3 lines,

$$x = a_1 - a_2 t,$$

and for $2p - 1$ lines, where $p \geqq 3$,

$$x = \frac{\begin{vmatrix} a_1 & \cdots & a_{p-1} \\ \cdot & \cdot & \cdot \\ a_{p-1} & \cdots & a_{2p-3} \end{vmatrix}}{\begin{vmatrix} a_3 & \cdots & a_p \\ \cdot & \cdot & \cdot \\ a_p & \cdots & a_{2p-3} \end{vmatrix}} - \frac{\begin{vmatrix} a_2 & \cdots & a_p \\ \cdot & \cdot & \cdot \\ a_p & \cdots & a_{2p-2} \end{vmatrix}}{\begin{vmatrix} a_3 & \cdots & a_p \\ \cdot & \cdot & \cdot \\ a_p & \cdots & a_{2p-3} \end{vmatrix}} t$$

and the Clifford point for $2p$ lines, $p \geqq 2$, is

$$x = \frac{\begin{vmatrix} a_1 & \cdots & a_p \\ \cdot & \cdot & \cdot \\ a_p & \cdots & a_{2p-1} \end{vmatrix}}{\begin{vmatrix} a_3 & \cdots & a_{p+1} \\ \cdot & \cdot & \cdot \\ a_{p+1} & \cdots & a_{2p-1} \end{vmatrix}}.$$

The expression for the circle fails to represent a circle when the numerator of the fraction representing the radius vanishes while the denominator does not,* and in all cases where the denominator vanishes; and the expression for the Clifford point similarly fails whenever the denominator vanishes. Sets of lines for which these expressions thus fail will be called improper, all other sets proper. A necessary and sufficient condition that a set of lines be improper is given in theorem 13, § 5. The facts for 3, 4, 5, and 6 lines are well known; but a brief consideration of these cases is necessary to indicate the method of treatment of the general case and to establish a basis for an inductive proof.

For 3 lines the expression represents a circle except when $a_2 = 0$; and this is the necessary and sufficient condition that the three lines be concurrent. They meet at the point a_1.

For 4 lines the Clifford point is

$$x_1 = \frac{\begin{vmatrix} a_1 & a_2 \\ a_2 & a_3 \end{vmatrix}}{a_3},$$

and this expression becomes meaningless when $a_3 = 0$. Since for 4 lines

* This case where the radius is zero is included among the cases of failure, first, because the Clifford circle is only defined by the Clifford points which lie on it, and if these points coincide they do not define a unique circle; and secondly, because of the close relationship of this case to the other cases of failure.

a_2 and a_3 are complementary, we have also $a_2 = 0$; so that both the numerator and denominator of the fraction vanish. This occurs when, and only when, the 4 lines are concurrent, meeting at the point a_1.

For 5 lines the Clifford circle is

$$x = \frac{\begin{vmatrix} a_1 & a_2 \\ a_2 & a_3 \end{vmatrix}}{a_3} - \frac{\begin{vmatrix} a_2 & a_3 \\ a_3 & a_4 \end{vmatrix}}{a_3} t$$

and failure occurs when

$$(1) \quad \begin{vmatrix} a_2 & a_3 \\ a_3 & a_4 \end{vmatrix} = 0, \qquad a_3 \neq 0;$$

$$(2) \quad a_3 = 0, \qquad \begin{vmatrix} a_2 & a_3 \\ a_3 & a_4 \end{vmatrix} \neq 0, \qquad \begin{vmatrix} a_1 & a_2 \\ a_2 & a_3 \end{vmatrix} \neq 0;$$

$$(3) \quad \begin{vmatrix} a_1 & a_2 \\ a_2 & a_3 \end{vmatrix} = \begin{vmatrix} a_2 & a_3 \\ a_3 & a_4 \end{vmatrix} = a_3 = 0.$$

Case (1), $\begin{vmatrix} a_2 & a_3 \\ a_3 & a_4 \end{vmatrix} = 0$, $a_3 \neq 0$. Since $\begin{vmatrix} a_2 & a_3 \\ a_3 & a_4 \end{vmatrix} = 0$, we must have

$$\begin{vmatrix} c_1 t_1^2 & c_1 t_1 & t_1^2 & t_1 & 1 \\ c_2 t_2^2 & c_2 t_2 & t_2^2 & t_2 & 1 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ c_5 t_5^2 & c_5 t_5 & t_5^2 & t_5 & 1 \end{vmatrix} = 0.$$

Not all of the minors of the elements in the first column can vanish. For these minors are the α_3's for the sets of 4 out of the 5 lines; and if they all vanished we would have $a_3 - a_4 t = 0$ for five different values of t, and hence it would follow that $a_3 = 0$, contrary to hypothesis. Suppose the non-vanishing minor to be in the upper right-hand corner. Then the reflex point c and clinant t of each of the 5 lines must satisfy the equation

$$\text{(I)} \quad A_1 ct^2 + A_2 ct + B_1 t^2 + B_2 t + B_3 = 0,$$

where the A's and B's are the co-factors of the elements of the lower row, and $A_1 \neq 0$. Solving for ct we have

$$ct = (-B_1 t^2 - B_2 t - B_3)/(A_1 t + A_2);$$

and multiplying both numerator and denominator by $1/t_1 t_2 t_3 t_4$, this takes the form

$$\text{(II)} \quad ct = (\alpha t^2 + rt + \bar{\alpha})/(\gamma t + \bar{\gamma}),$$

where r is real and $\gamma \neq 0$. Since the c and t for each of the 5 lines satisfy (I), they must also satisfy (II) unless the value of t is such as to cause the denominator $\gamma t + \bar{\gamma}$ to vanish. If this is true for any t, this t must also

cause the numerator to vanish; and in this case the denominator would be a factor of the numerator. If then in equation (II) the denominator is not a factor of the numerator, this equation is satisfied by all of the 5 lines; and the 5 lines are evidently tangents of the parabola whose envelope equation is

$$tx + y = (\alpha t^2 + rt + \bar{\alpha})/z\gamma t + \bar{\gamma}).$$

If, on the other hand, $\gamma t + \bar{\gamma}$ is factor of the numerator, we may write

$$\text{(III)}\ ct = \alpha' t + \bar{\alpha}',$$

and this equation must b satisfied by each of the 5 lines except possibly one whose clinant makes $\gamma t + \bar{\gamma}$ vanish. But all lines satisfying (III) are concurrent, and not all of our 5 lines can be concurrent since $a_3 \neq 0$. Hence 4, and only 4, of the 5 lines are concurrent.

If, then, $\begin{vmatrix} a_2 & a_3 \\ a_3 & a_4 \end{vmatrix} = 0$ and $a_3 \neq 0$, either the 5 lines touch a parabola or just 4 of them are concurrent; and, conversely, if 5 lines touch a parabola or just 4 of them are concurrent, then $\begin{vmatrix} a_2 & a_3 \\ a_3 & a_4 \end{vmatrix} = 0$ and $a_3 \neq 0$.

Case (2), $a_3 = 0$, $\begin{vmatrix} a_2 & a_3 \\ a_3 & a_4 \end{vmatrix} \neq 0$, $\begin{vmatrix} a_1 & a_2 \\ a_2 & a_3 \end{vmatrix} \neq 0$. Obviously $a_2 \neq 0$ and $a_4 \neq 0$; and $a_3 = 0$ gives

$$\begin{vmatrix} c_1 t_1^2 & t_1^3 & t_1^2 & t_1 & 1 \\ c_2 t_2^2 & t_2^3 & t_2^2 & t_2 & 1 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ c_5 t_5^2 & t_5^3 & t_5^2 & t_5 & 1 \end{vmatrix} = 0.$$

Hence each of the 5 lines satisfies the equation

$$Act^2 + B_1 t^3 + B_2 t^2 + B_3 t + B_4 = 0,$$

where the A and the B's are co-factors of the elements of the lower row, and $A \neq 0$. Dividing through by $1/(t_1 t_2 t_3 t_4)^{3/2}$, and solving for ct, we have

$$ct = (\alpha_1 t^3 + \alpha_2 t^2 + \bar{\alpha}_2 t + \bar{\alpha}_1)/(rt),$$

where r is real and $\neq 0$. Since $t \neq 0$ for any line, all of the 5 lines must satisfy this equation. Moreover, we can not have $\alpha_1 = 0$; for this would give $ct = \alpha' t + \bar{\alpha}'$, which would make the 5 lines concurrent, contrary to the condition $a_2 \neq 0$. Hence all of the 5 lines must touch the deltoid whose envelope equation is

$$tx + y = (\alpha_1 t^3 + \alpha_2 t^2 + \bar{\alpha}_2 t + \bar{\alpha}_1)/(rt).$$

Conversely, if 5 lines touch a deltoid we must have $a_3 = 0$ and

$$\begin{vmatrix} a_2 & a_3 \\ a_3 & a_4 \end{vmatrix} \neq 0, \qquad \begin{vmatrix} a_1 & a_2 \\ a_2 & a_3 \end{vmatrix} \neq 0.$$

Case (3), $\begin{vmatrix} a_1 & a_2 \\ a_2 & a_3 \end{vmatrix} = \begin{vmatrix} a_2 & a_3 \\ a_3 & a_4 \end{vmatrix} = a_3 = 0$. It follows at once that $a_2 = a_4 = 0$, and hence that the 5 lines are concurrent at the point a_1.

For 6 lines the Clifford point is

$$x_1 = \begin{vmatrix} a_1 & a_2 & a_3 \\ a_2 & a_3 & a_4 \\ a_3 & a_4 & a_5 \end{vmatrix} \Bigg/ \begin{vmatrix} a_3 & a_4 \\ a_4 & a_5 \end{vmatrix}$$

and failure occurs when

$$(1) \quad \begin{vmatrix} a_3 & a_4 \\ a_4 & a_5 \end{vmatrix} = 0, \quad \begin{vmatrix} a_1 & a_2 & a_3 \\ a_2 & a_3 & a_4 \\ a_4 & a_5 & a \end{vmatrix} \neq 0;$$

$$(2) \quad \begin{vmatrix} a_3 & a_4 \\ a_4 & a_5 \end{vmatrix} = \begin{vmatrix} a_1 & a_2 & a_3 \\ a_2 & a_3 & a_4 \\ a_3 & a_4 & a_5 \end{vmatrix} = 0.$$

Case (1), $\begin{vmatrix} a_3 & a_4 \\ a_4 & a_5 \end{vmatrix} = 0$, $\begin{vmatrix} a_1 & a_2 & a_3 \\ a_2 & a_3 & a_4 \\ a_3 & a_4 & a_5 \end{vmatrix} \neq 0$. With $\begin{vmatrix} a_3 & a_4 \\ a_4 & a_5 \end{vmatrix} = 0$ we have the complementary condition $\begin{vmatrix} a_2 & a_3 \\ a_3 & a_4 \end{vmatrix} = 0$. Then unless $a_3 = a_4 = 0$, all the determinants of the 2nd order in the matrix $\begin{Vmatrix} a_2 & a_3 & a_4 \\ a_3 & a_4 & a_5 \end{Vmatrix}$ would vanish, causing $\begin{vmatrix} a_1 & a_2 & a_3 \\ a_2 & a_3 & a_4 \\ a_3 & a_4 & a_5 \end{vmatrix}$ to vanish. It follows that $a_3 = a_4 = 0$, and therefore also that $a_2 \neq 0$, $a_5 \neq 0$. Since $a_3 - a_4 t = 0$ for all values of t, we have $\alpha_3 = 0$ for every set of 5 put of the 6 lines. Moreover, $\begin{vmatrix} \alpha_2 & \alpha_3 \\ \alpha_3 & \alpha_4 \end{vmatrix}$ can not vanish for any set of 5; for if it did, it would follow for that set of 5 lines that $\alpha_2 = \alpha_3 = \alpha_4 = 0$ or, in terms of the a's for the 6 lines, $a_2 - a_3 t = a_3 - a_4 t = a_4 - a_5 t = 0$, contrary to the condition, $\begin{vmatrix} a_1 & a_2 & a_3 \\ a_2 & a_3 & a_4 \\ a_3 & a_4 & a_5 \end{vmatrix} \neq 0$. Hence each set of 5 out of the 6 lines touch a deltoid; and since a deltoid is uniquely determined by four of its tangents, it follows that the 6 lines all touch one deltoid. And conversely, if 6 lines touch a deltoid, we must have $a_3 = a_4 = 0$, and $a_2 \neq 0$, $a_5 \neq 0$.

Case (2), $\begin{vmatrix} a_3 & a_4 \\ a_4 & a_5 \end{vmatrix} = \begin{vmatrix} a_1 & a_2 & a_3 \\ a_2 & a_3 & a_4 \\ a_3 & a_4 & a_5 \end{vmatrix} = 0$. As in case (1), $\begin{vmatrix} a_2 & a_3 \\ a_3 & a_4 \end{vmatrix} = 0$

also. If $a_3 = 0$, it follows that $a_4 = a_2 = a_5 = 0$, and hence that the 6 lines are concurrent. If on the contrary $a_3 \neq 0$, then $a_4 \neq 0$, $a_2 \neq 0$, $a_5 \neq 0$, and all the determinants of the 2nd order in the matrix $\begin{Vmatrix} a_2 & a_3 & a_4 \\ a_3 & a_4 & a_5 \end{Vmatrix}$ vanish. Since then $\begin{vmatrix} a_2 & a_3 & a_4 \\ a_3 & a_4 & a_5 \\ t^2 & t & 1 \end{vmatrix} = 0$ for all values of t, we have $\begin{vmatrix} \alpha_2 & \alpha_3 \\ \alpha_3 & \alpha_4 \end{vmatrix} = 0$ for every set of 5 out of the 6 lines. Also α_3 can not vanish for more than one set of 5. There are then two possibilities in this case, viz., either all of the 6 lines touch a parabola, or just 5 of them are concurrent. And conversely, if 6 lines touch a deltoid, or if just 5 of them are concurrent, then all the determinants of 2nd order in the matrix $\begin{Vmatrix} a_2 & a_3 & a_4 \\ a_3 & a_4 & a_5 \end{Vmatrix}$ vanish, and $a_2 \neq 0$, $a_3 \neq 0$, $a_4 \neq 0$, $a_5 \neq 0$.

§ 3. *The Curves $P_k^{(m)}$.*

A polynomial of the form

$$\alpha_1 t^n + \alpha_2 t^{n-1} + \cdots + \bar{\alpha}_2 t + \bar{\alpha}_1 \equiv P(t),$$

in which the coefficients of t^i and t^{n-i} are conjugates, may be called a *self-conjugate polynomial of degree n in t.* If n is even, the coefficient of $t^{n/2}$ is real. Such a polynomial has a theory analogous to that of a polynomial with real coefficients. Expressing it as a product of linear factors,

$$P(t) \equiv \alpha_1 (t - \rho_1)(t - \rho_2) \cdots (t - \rho_n),$$

all of the ρ's which are not turns must pair off into inverse pairs, *i.e.*, into pairs ρ_1 and ρ_2 such that $\rho_1 \bar{\rho}_2 = \bar{\rho}_1 \rho_2 = 1$. Thus the number of non-turn roots of $P(t) = 0$ is always even; and if n is odd, at least one root must be a turn. Any such self-conjugate polynomial may be decomposed into self-conjugate polynomial factors, no factor being of degree higher than the second. If a self-conjugate polynomial is divisible by another self-conjugate polynomial, the quotient is also self-conjugate; and the highest common factor of two self-conjugate polynomials is a self-conjugate polynomial. It will be convenient to consider

$$\alpha_{i+1} t^{n-i} + \alpha_{i+2} t^{n-i-1} + \cdots + \bar{\alpha}_{i+2} t^{i+1} + \bar{\alpha}_{i+1} t^i$$

as a self-conjugate polynomial *of degree n.* It may be understood that 0 and ∞ are i-fold inverse roots in this case. Thus $t^3 + t^2$ is a self-conjugate polynomial of degree 5 rather than of degree 3. In breaking this polynomial up into self-conjugate factors, the factor t is to be regarded as a self-conjugate polynomial of degree 2.

The notation $P_k^{(m)}$ will indicate the curve whose envelope equation is

$$tx + y = \frac{\alpha_1 t^m + \alpha_2 t^{m-1} + \cdots + \bar{\alpha}_2 t + \bar{\alpha}_1}{\gamma_{k+1} t^{m-k-1} + \gamma_{k+2} t^{m-k} + \cdots + \bar{\gamma}_{k+2} t^{k+1} + \gamma_{k+1} t^k} \equiv \frac{\alpha(t)}{\gamma(t)}$$

where $m \geqq 1$; $0 \leqq k \leqq (m-1)/2$; $\gamma_{k+1} \neq 0$; and $\alpha(t)$ and $\gamma(t)$ are self-conjugate polynomials of degree m and $m-1$ respectively, with no common polynomial factor. From this last restriction it follows that for $k > 0$, $\alpha_1 \neq 0$; and it may also be understood to imply that $\alpha(t)$ can not vanish identically except in the rather trivial case $m = 1$, $k = 0$. The map equation of this curve is

$$x = \frac{d}{dt}\left[\frac{\alpha(t)}{\gamma(t)}\right] = \frac{\gamma(t)\alpha'(t) - \alpha(t)\gamma'(t)}{\gamma^2(t)}.$$

It is a rational curve of class m. Assuming for the moment the projective point of view, we may say that the curve is tangent to the line at infinity in the $m-1$ directions given by the roots of $\gamma(t) = 0$ (real directions for roots that are turns, pairs of imaginary directions for the pairs of inverse roots). If μ is the multiplicity of a root of $\gamma(t) = 0$, the curve has contact of the μth order in the corresponding direction. In particular, corresponding to the roots 0 and ∞, there is contact of the kth order at the circular points I and J. If σ represents the number of *distinct* roots of $\gamma(t) = 0$, the curve is of order $m + \sigma - 1$; the line at infinity counts as

$$\tfrac{1}{2}[(m-1)(m-4) + 2\sigma]$$

double tangents and as $m - \sigma - 1$ flex tangents, thus accounting for all the line singularities. There are $m + 2\sigma - 4$ cusps and

$$\tfrac{1}{2}[(m+\sigma)^2 - 7m - 9\sigma + 14]$$

nodes, these cusps and nodes not being necessarily distinct. Two such curves with the same values for m and k have m^2 common tangents, of which the line at infinity counts for at least* $(m-1)^2 + 2k$. Hence they have, aside from the line at infinity, not more than $2m - 2k - 1$ common tangents; and therefore such a curve is uniquely determined by $2m - 2k$ of its tangents. Two curves $P_k^{(m_1)}$ and $P_k^{(m_2)}$ have $m_1 m_2$ common tangents, of which the line at infinity counts for at least $(m_1 - 1)(m_2 - 1) + 2k$. Hence they have, aside from the line at infinity, not more than $m_1 + m_2 - 2k - 1$ common tangents.

This curve $P_k^{(m)}$ is the most general type of curve that can be represented by an envelope equation. It is the dual of the Jonquieres type,

* The line at infinity counts for more than $(m-1)^2 + 2k$ of the common tangents in case the equations $\gamma(t) = 0$ for the two curves have common roots other than 0 and ∞.

since it is rational and the line at infinity furnishes all of its line singularities. For $k = 0$ it is the $(m-1)$-fold parabola* of Clifford's paper. As special cases, it may be noted that a $P_0^{(1)}$ is merely a point, a $P_0^{(2)}$ is an ordinary quadratic parabola, and a $P_1^{(3)}$ is a deltoid. The curve $\Delta^{(2n-1)}$ of Morley† and Stephens‡ is a $P_{n-1}^{(2n-1)}$; and the curve $K^{(2n)}$ of Atchison§ is a $P_{n-1}^{(2n)}$.

§ 4. *Characteristic Matrices for N Lines.*

Let the positive integer p be defined by the relations $n = 2p - 1$ when n is odd and $n = 2p$ when n is even; and let $[h, i]$, where $h \geqq 1$, $i \leqq p$, $i - h \geqq 0$, represent the matrix||

$$\left\| \begin{matrix} a_h & a_{h+1} & \cdots & a_{n-i+1} \\ a_{h+1} & a_{h+2} & \cdots & a_{n-i+2} \\ \cdot & \cdot & \cdot & \cdot \\ a_i & a_{i+1} & \cdots & a_{n-h+1} \end{matrix} \right\|.$$

For a set of n lines there is a triangular table of such characteristic matrices, as follows:

$$\begin{matrix} [1,1] \\ [1,2] & [2,2] \\ \vdots & \vdots \\ [1,i] & [2,i] & \cdots & [h,i] & \cdots & [i,i] \\ \vdots & \vdots & & \vdots \\ [1,p-1] & [2,p-1] & \cdots & [h,p-1] & \cdots & [p-1,p-1] \\ [1,p] & [2,p] & \cdots & [h,p] & \cdots & [p-1,p] & [p,p]. \end{matrix}$$

In this table, a constant h gives a column, a constant i gives a row, a constant difference $i - h$ gives a principal diagonal (running downwards to the right), and a constant sum $i + h$ gives a skew diagonal (upwards to the right). For $n = 2p - 1$ the difference between the number of columns and the number of rows in each matrix is even, and the matrices of the lower row are square; while for $n = 2p$ this difference is odd, and there are no square matrices. All the matrices are self-complementary in the sense that two elements symmetrically situated with respect to the center are complementary.

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* Clifford does not state explicitly that his multi-fold parabola must not be tangent to the line at infinity at the circular points, but he ignores the consequences which would result from such specialization of his curves.

† "Orthocentric Properties of the Plane N-Line," *Trans. of the Amer. Math. Soc.*, Vol. 4, page 1; 1903.

‡ "On the Pentadeltoid," *Trans. of the Amer. Math. Soc.*, Vol. 7, page 207; 1906.

§ "Curves with a Directrix," Johns Hopkins dissertation, 1908.

|| This matrix is also a function of n, but it is simpler to omit the letter n from the symbol.

We shall be concerned with the vanishing of the determinants of highest order in these matrices, and the following notation will be used:

$[h, i] = 0$ means that every determinant of highest order vanishes;

$[h, i] \neq 0$ means that at least one determinant of highest order does not vanish;

$\{[h, i]\} = 0$ means that every solid* determinant of highest order vanishes;

$\{[h, i]\} \# 0$ means that no solid determinant of highest order vanishes;

$\}[h, i]\{ = 0$ means that at least one solid determinant of highest order vanishes;

$\{[h, i]\} \doteq 0$ means that all the solid determinants of the matrix vanish except the first and the last which do not vanish.

The two following well-known theorems will be used:

THEOREM A.† If D is a determinant of order m, A_{11}, A_{1m}, A_{m1}, and A_{mm} are the first minors of the four corner elements, and K is the second minor obtained by striking off a border one element wide all around; then

$$DK = \begin{vmatrix} A_{mm} & A_{m1} \\ A_{1m} & A_{11} \end{vmatrix}.$$

THEOREM B.‡ If in a matrix a determinant of order r does not vanish, and if every determinant of order $r + k$ which contains this non-vanishing determinant vanishes; then every determinant of order $r + k$ in the matrix vanishes.

THEOREM 1. If $[h, i] = 0$, then also $[h', i'] = 0$ for any h' and i' such that $i' + h' \geqq i + h$ and $i' - h' \geqq i - h$.

These matrices $[h', i']$ cover an area in the table of matrices which is an isosceles triangle with vertex at the matrix $[h, i]$ and base in the lowest row, or such portion of such a triangle as is not cut off by the left-hand boundary of the table. The truth of the theorem is obvious from the fact that the matrix $[h, i]$ furnishes a band extending horizontally across the matrix $[h', i']$.

THEOREM 2. If $[h, i] = 0$, then $\{[h', i']\} = 0$ for any h' and i' such that $i - h \leqq i' - h' \leqq n - h - i + 1$ and $i' \geqq i$.

The matrices $[h', i']$ of this theorem make up a parallelogram with the matrix $[h, i]$ at one vertex and its base in the lowest row, or such portion of such a parallelogram as is not cut off by the left hand boundary of the table. (For $h = 1$ this theorem adds no information to that given by theorem 1.) The proof consists in the fact that the matrix $[h, i]$ furnishes

* A solid determinant of highest order means one made up of consecutive columns of the matrix.

† Bôcher, "Higher Algebra," page 33.

‡ This theorem is a slight extension of the one given by Bôcher, "Higher Algebra," page 54.

a band extending horizontally across any solid determinant of the matrix $[h', i']$.

THEOREM 3. If $\{[h, i]\} = 0$ and $i - h \geqq 1$, then either $\{[h+1, i]\} = 0$ or $\{[h+1, i]\} \neq 0$.

The condition $i - h \geqq 1$ is needed in order that the matrix $[h+1, i]$ should exist. For $i = p$, $n = 2p - 1$, the theorem is trivial. For $i = p$, $n = 2p$, it follows from the fact that the two solid determinants in the matrix $[h+1, i]$ are complementary and must vanish together. For $i < p$, Theorem A shows that any three successive solid determinants of the matrix $[h, i-1]$ are such that the square of the middle one is equal to the product of the two adjacent ones. It follows at once that either $\{[h+1, i]\} = 0$ or $\{[h+1, i]\} \neq 0$.

The notation $\{[h, i]\} \neq 0*$ will be used to indicate that either $\{[h, i]\} \neq 0$ or else $h > i$ and therefore the matrix $[h, i]$ does not exist.

THEOREM 4. If $\{[h, i]\} = 0$ and $\{[h+1, i]\} \neq 0*$, then $\{[h, i-1]\} \neq 0*$ and $[h, i] = 0$.

If $h = i$, the matrix $[h+1, i]$ does not exist, and also the matrix $[h, i-1]$ does not exist. Also, in this case, there will be only one row in the matrix $[h, i]$, and hence the statements $\{[h, i]\} = 0$ and $[h, i] = 0$ are identical.

If $i > h$, then $\{[h+1, i]\} \neq 0$, and from the argument for the preceding theorem it follows that $\{[h, i-1]\} \neq 0$. Then the statement $[h, i] = 0$ will follow from $\{[h, i]\} = 0$ and $\{[h+1, i]\} \neq 0$ by a repeated use of Theorem B.

THEOREM 5. If $[h, i] = 0$ and $\{[h+1, i]\} \neq 0*$, and

$$\}[h+i+j-n-2, j]\{ = 0$$

where $j \geqq i - 1$; then $[h-1, i-1] = 0$.

From $[h, i] = 0$ it follows that $\{[h', i']\} = 0$ over a parallelogram area as given in Theorem 2. The matrix $[h+i+j-n-2, j]$ is in the principal diagonal immediately to the left of this parallelogram. The existence of this matrix gives us the additional inequality $h + i + j - n - 2 \geqq 1$. All the solid determinants of $[h-1, i-1]$ except possibly the first and last (which are complementary), vanish because of the condition $[h, i] = 0$. To prove therefore that $\{[h-1, i-1]\} = 0$ it is only necessary to show that either the first or last solid determinant in the matrix vanishes.

Let

$$\begin{vmatrix} a_{h+i+j-n-2+\lambda} & \cdots & a_{j+\lambda} \\ \cdot\ \cdot\ \cdot & \cdot\ \cdot\ \cdot & \cdot\ \cdot\ \cdot \\ a_{j+\lambda} & \cdots & a_{j+n-h-i+2+\lambda} \end{vmatrix}$$

where $0 \leqq \lambda \leqq n - 2j + 1$, be the vanishing solid determinant of the

matrix $[h+i+j-n-2,\ j]$; and consider first the case where $i = h$ and therefore $[h+1,\ i]$ does not exist. The condition $[h,\ i] = 0$ in this case is simply $a_h = a_{h+1} = \cdots = a_{n-h+1} = 0$; and the vanishing determinant takes the form

$$\begin{vmatrix} a_{h+i+j-n-2+\lambda} & \cdots & a_{h-1} & 0 & \cdots & 0 \\ a_{h+i+j-n-1+\lambda} & \cdots & 0 & 0 & \cdots & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ a_{h-1} & \cdots & 0 & 0 & \cdots & 0 \\ 0 & \cdots & 0 & 0 & \cdots & a_{n-h+2} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & \cdots & 0 & a_{n-h+2} & \cdots & a_{j+n-h-i+2+\lambda} \end{vmatrix}$$

or $a_{h-1}^r a_{n-h+2}^s = 0$, where $r = n - i - j + 1 - \lambda$ and $s = j - i + 2 + \lambda$. Since a_{h-1} and a_{n-h+2} are complementary, it follows that they both vanish, and that therefore $\{[h-1,\ i-1]\} = 0$; and this is the same, in this case, as $[h-1,\ i-1] = 0$.

Suppose next that $i - h > 0$, and that therefore $[h+1,\ i]$ exists. The vanishing solid determinant of $[h+i+j-n-2,\ j]$ may be written in the form

$$\begin{vmatrix} a_{h+i+j-n-2+\lambda} & \cdots & a_{h-1} & a_h & \cdots & \cdots & \cdots & a_{j+\lambda} \\ a_{h+i+j-n-1+\lambda} & \cdots & a_h & a_{h+1} & \cdots & \cdots & \cdots & a_{j+\lambda+1} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ a_{h-1} & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & a_{n-i+1} \\ a_h & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & a_{n-i+2} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ a_{j-i+h+\lambda} & \cdots & \cdots & \cdots & \cdots & a_{n-i+1} & \cdots & \cdots \\ a_{j-i+h+\lambda+1} & \cdots & \cdots & a_{n-2i+h+3} & \cdots & a_{n-i+2} & \cdots & \cdots \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ a_{j+\lambda} & \cdots & \cdots & a_{n-i+2} & \cdots & a_{n-h+1} & \cdots & a_{j+n-h-i+2+\lambda} \end{vmatrix}$$

From the fact that $[h,\ i]$ is of rank $i - h$, it follows that there is a (homogeneously) unique set of constants $\lambda_1, \lambda_2, \cdots, \lambda_{i-h+1}$ such that if the elements of each row of this matrix $[h, i]$ be multiplied by the corresponding λ, the sum of each column will then be zero; and one may use for the λ's the $i - h + 1$ determinants of order $i - h$ in the first $i - h$ columns. If then in the above determinant we multiply the first $i - h + 1$ top rows respectively by these λ's, and add for a new top row; and do the same for each successive block of $i - h + 1$ rows down to the lowest block of $i - h + 1$ rows; we obtain the following vanishing determinant

$$\begin{vmatrix} \cdots & \cdots & D_1 & 0 & \cdots & 0 & \cdots & \cdots & 0 \\ \cdots & \cdots & 0 & 0 & \cdots & 0 & \cdots & \cdots & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ D_1 & \cdots & 0 & 0 & \cdots & 0 & \cdots & \cdots & 0 \\ 0 & \cdots & 0 & 0 & \cdots & 0 & \cdots & \cdots & D_2 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & \cdots & 0 & 0 & \cdots & 0 & D_2 & \cdots & \cdots \\ a_{j-i+h+\lambda+1} & \cdots & \cdots & a_{n-2i+h+3} & \cdots & a_{n-i+2} & \cdots & \cdots & \cdots \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ a_{j+\lambda} & \cdots & \cdots & a_{n-i+2} & \cdots & a_{n-h+1} & \cdots & \cdots & a_{j+n-h-i+2+\lambda} \end{vmatrix} = 0;$$

or $D_1^r D_2^s D_3 = 0$, where

$$D_1 = \begin{vmatrix} h-1 & \cdots & i-1 \\ \cdots & \cdots & \cdots \\ i-1 & \cdots & 2i-h-1 \end{vmatrix},$$

$$D_2 = \begin{vmatrix} n-i+2 & h & h+1 & \cdots & i-1 \\ \cdots & \cdots & \cdots & & \cdots \\ n-h+2 & i & i+1 & \cdots & 2i-h-1 \end{vmatrix},$$

and

$$D_3 = \begin{vmatrix} n-2i+h+3 & \cdots & n-i+2 \\ \cdots & & \cdots \\ n-i+2 & \cdots & n-h+1 \end{vmatrix}.$$

$D_3 \neq 0$; for it is the last solid determinant in $[h, i-1]$, and from Theorem 4, $\{[h, i-1]\} \# 0$. Hence either D_1 or D_2 must vanish. D_1 is the first solid determinant in $[h-1, i-1]$; and if $D_2 = 0$, by the repeated use of Theorem B, with the hypothesis $[h, i] = 0$, it will follow that the last solid determinant in $[h-1, i-1]$ vanishes. This completes the proof that $\{[h-1, i-1]\} = 0$. But with $\{[h, i-1]\} \# 0$, Theorem 4 gives $[h-1, i-1] = 0$.

Theorem 6. If $[h, i] = 0$ and $\{[h+1, i]\} \# 0 \ast$ and

$$[h+i+i'-1, i'] = 0,$$

where $i' \gtreqless i$; then $[h-1, i-1] = 0$.

The matrix $[h+i-i'-1, i']$ is any matrix in the skew diagonal immediately to the left of the matrix $[h, i]$. By Theorem 2,

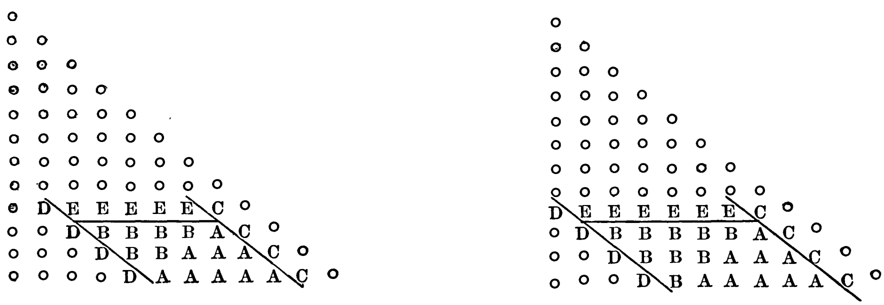
$$[h+i-i'-1, i'] = 0$$

gives $\{[h+i+i'-n-2, i']\} = 0$. This, with $[h, i] = 0$ and $\{[h+1, i]\} \# 0 \ast$, satisfies the hypothesis of Theorem 5, and hence gives $[h-1, i-1] = 0$.

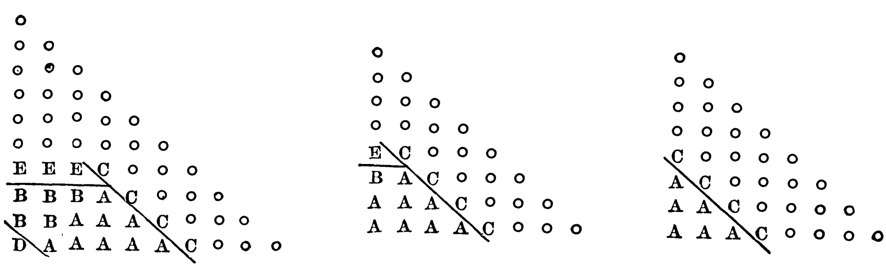
By means of the six foregoing theorems, it may be shown that any characteristic matrix for n lines which has the property $[h, i] = 0$ or

$\{[h, i]\} = 0$ lies in a "vanishing area" in the triangular table of matrices; this vanishing area being a parallelogram or portion of a parallelogram. Every such vanishing parallelogram extends downwards to include the lowest (or pth) row; and is made up of matrices of two kinds A and B, as shown in the figure below, having the respective properties $[A] = 0$ and $\{[B]\} = 0$. For n odd, the matrices in the upper right and lower left corners of the parallelogram are both A's, and lie on the same skew diagonal. For n even, the upper right matrix is an A while the lower left is a B, and the skew diagonal containing the former is immediately to the right of that containing the latter. This vanishing parallelogram is bordered on three sides by a non-vanishing border of matrices C, D, and E, having the properties respectively $\{[C]\} \# 0 *$, $\{[D]\} \# 0$, $\{[E]\} \doteq 0$.

A vanishing area may be only part of such a parallelogram, a part at the left being cut off by the left-hand boundary of the table, as shown in the following figures:



Vanishing parallelogram for n odd.



Vanishing parallelogram for n even.

Because of the non-vanishing border, two vanishing areas can obviously not overlap.

The matrix $[h, i]$, in which $h \geqq 2$, will be called a *key matrix*

(a) when $h = 2$, if $[h, i] = 0$ and $\{[h + 1, i]\} \# 0 *$; and

(b) when $h > 2$, if $[h, i] = 0$, $\{[h + 1, i]\} \# 0 *$, and $[h - 1, i] \neq 0$.

From this definition it may be seen that there is one and only one key

matrix in every vanishing area.* It is the single matrix A in the top row of the vanishing area, provided that this matrix is not in the first column. When the matrix A in the top row of a vanishing area is in the first column, the key matrix is then the matrix A in the row next below and the second column. If $[2, i]$ is a key matrix, we may or may not have $[1, i-1] = 0$.

THEOREM 7. If $[h, i] = 0$ for a set of $n-1$ out of n lines, then $[h, i+1] = 0$ for the set of n lines.

In the matrix

$$\begin{Vmatrix} \alpha_h & \cdots & \alpha_{n-i} \\ \cdot & \cdot\;\cdot\;\cdot & \cdot \\ \alpha_i & \cdots & \alpha_{n-h} \end{Vmatrix}$$

for the $n-1$ lines, all the determinants of highest order vanish. Hence in the matrix

$$\begin{Vmatrix} a_h & \cdots & a_{n-i} & t^{i-h+1} \\ \cdot & \cdot\;\cdot\;\cdot & \cdot & \cdot \\ a_i & \cdots & a_{n-h} & t \\ a_{i+1} & \cdots & a_{n-h+1} & 1 \end{Vmatrix}$$

where the a's are for the n lines, all the determinants of highest order which contain the last column vanish. It follows, by Theorem B, that all the determinants of highest order in the matrix vanish, and hence that $[h, i+1] = 0$ for the n lines.

THEOREM 8. If $[h, i] = 0$ for n lines, and $i \leqq n-p$; then $\{[h, i]\} = 0$ for any set of $n-1$ out of the n lines.

The condition $i \leqq n-p$ is needed to insure the existence of the matrix $[h, i]$ for $n-1$ lines; for $i > n-p$ can only hold when n is odd and $i = p$, in which case the matrix $[h, i]$ would be in the lowest row of the table for n lines, and would not exist for $n-1$ lines.

In the matrix

$$\begin{Vmatrix} a_h & \cdots & a_{n-i+1} \\ \cdot & \cdot\;\cdot\;\cdot & \cdot \\ a_i & \cdots & a_{n-h+1} \end{Vmatrix}$$

all the determinants of highest order vanish. Hence all the determinants of highest order in the matrix

$$\begin{Vmatrix} a_h & \cdots & \cdots & a_{n-i+1} \\ \cdot & \cdot\;\cdot\;\cdot & \cdot\;\cdot\;\cdot & \cdot \\ a_i & \cdots & \cdots & a_{n-h+1} \\ t^{n-h-i+1} & \cdots & t & 1 \end{Vmatrix}$$

vanish for all values of t. But when t is the clinant of one of the n lines,

* A trivial exception which will not concern us is the vanishing area in which the only matrix A is the matrix $[1, p]$.

the solid determinants of this matrix are (except for a factor which is a power of t) the solid determinants of the matrix

$$\left\|\begin{matrix} \alpha_h & \cdots & \alpha_{n-i} \\ \cdot & \cdot & \cdot \\ \alpha_i & \cdots & \alpha_{n-h} \end{matrix}\right\|$$

where the α's are for the remaining $n-1$ lines. Hence $\{[h, i]\} = 0$ for any set of $n-1$ out of the n lines.

THEOREM 9. If $[h, i]$ is a key matrix for n lines, where $i \gtreqless n - p$, then $[h, i]$ is also a key matrix for at least $n + h - i$ of the sets of $n-1$ out of the n lines.

By the preceding theorem, $\{[h, i]\} = 0$ for any set of $n-1$ out of the n lines. Hence for one of these sets of $n-1$ lines (see Theorem 3). either $\{[h+1, i]\} = 0$, or $\{[h+1, i]\} \# 0$, or $h = i$ and $[h+1, i]$ does not exist. Suppose that $\{[h+1, i]\} = 0$ for a certain set of $n-1$ lines, Then

$$\begin{vmatrix} \alpha_{h+1} & \cdots & \alpha_i \\ \cdot & \cdot & \cdot \\ \alpha_i & \cdots & \alpha_{2i-h+1} \end{vmatrix} = 0$$

for this set, or

$$\begin{vmatrix} a_{h+1} & \cdots & \cdot & a_{i+1} \\ \cdot & \cdot & \cdot & \cdot \\ a_i & \cdots & \cdot & a_{2i-h} \\ t^{i-h} & \cdots & t & 1 \end{vmatrix} = 0$$

where the a's are for the n lines and t is the clinant of the omitted line. If this last equation held for more than $i - h$ distinct values of t, it would hold for all values of t, and we would have

$$\begin{vmatrix} a_{h+1} & \cdots & a_i \\ \cdot & \cdot & \cdot \\ a_i & \cdots & a_{2i-h-1} \end{vmatrix} = 0.$$

But this is the first solid determinant in the matrix $[h+1, i]$ for n lines, and by hypothesis $\{[h+1, i]\} \# 0$. Hence we can not have $\{[h+1, i]\} = 0$ for more than $i - h$ sets of $n-1$ lines, and therefore we must have $\{[h+1, i]\} \# 0 *$ for at least $n + h - i$ sets. But if for a set of $n-1$ lines $\{[h, i]\} = 0$ and $\{[h+1, i]\} \# 0 *$, then, by Theorem 4, $[h, i] = 0$. For $h = 2$ this proves our theorem. For $h > 2$, it remains to show that for a set of $n-1$ lines for which $[h, i] = 0$ and $\{[h+1, i]\} \# 0 *$, we have also $[h-1, i] \neq 0$. Suppose that $[h-1, i] = 0$. Then, by Theorem 5, $[h-1, i-1] = 0$. Then, by Theorem 7, $[h-1, i] = 0$ for the n lines, which contradicts the hypothesis that $[h, i]$ is a key matrix for the n lines. Hence for our set of $n-1$ lines, $[h-1, i] \neq 0$, and $[h, i]$ is a key matrix.

THEOREM 10. If $[h, i]$ is a key matrix for all sets of $n-1$ out of n lines, then $[h, i]=0$ for the n lines.

In the matrix

$$\left\| \begin{matrix} \alpha_h & \cdots & \alpha_{n-i} \\ \cdot & \cdot & \cdot \\ \alpha_i & \cdots & \alpha_{n-h} \end{matrix} \right\|$$

for any one of the sets of $n-1$ lines, all the determinants of highest order vanish. Therefore in the matrix

$$\left\| \begin{matrix} a_h & \cdots & \cdot & a_{n-i+1} \\ \cdot & \cdot & \cdot & \cdot \\ a_i & \cdots & \cdot & a_{n-h+1} \\ t^{n-h-i+1} & \cdots & t & 1 \end{matrix} \right\|$$

where the a's are for the n lines and t is the clinant of any one of them, all the solid determinants vanish. Each one of these solid determinants is of the $(i-h+1)$th degree in t (except for a factor which is a power of t); and since $i-h+1<n$, it must vanish for all values of t. Hence all the solid determinants in

$$\left\| \begin{matrix} a_h & \cdots & a_{n-i+1} \\ \cdot & \cdot & \cdot \\ a_i & \cdots & a_{n-h+1} \end{matrix} \right\|$$

vanish, or $\{[h, i]\}=0$ for the n lines. Then either $\{[h+1, i]\}=0$ or $\{[h+1, i]\} \neq 0$*. Suppose $\{[h+1, i]\}=0$. Then $[h+1, i]$ would lie in a vanishing area having a certain key matrix; and, by the preceding theorem, some sets of $n-1$ lines would have the same key matrix; and therefore for some sets of $n-1$ lines we would have $\{[h+1, i]\}=0$. But this is contrary to the hypothesis that $[h, i]$ is a key matrix for all sets of $n-1$ lines. Therefore $\{[h+1, i]\} \neq 0$* for the n lines, and, by Theorem 4, it follows that $[h, i]=0$ for the n lines.

§ 5. *Necessary and Sufficient Condition for an Improper Set.*

THEOREM 11. (A) If $[k+2, i]$ is a key matrix for n lines, $n \geqq 3$, $0 \leqq k \leqq p-2$, $k+2 \leqq i \leqq p$; then there exists a unique integer g in the interval $n+k-i+2 \leqq g \leqq p-2$, such that g (and no more) of the n lines touch a unique curve $P_k^{(g-n+k+i-1)}$: and conversely,

(B) If g (and no more) of n lines touch a curve $P_k^{(g-n+k+i-1)}$, $n \geqq 3$, $0 \leqq k \leqq p-2$, $k+2 \leqq i \leqq p$, $n+k-i+2 \leqq g \leqq n$; then $[k+2, i]$ is a key matrix for the n lines.

It has been shown in § 2 that this theorem holds for $n=3, 4, 5, 6$; and the general theorem may be proved by induction. It will first be shown that (A) holds for n lines if (A) and (B) both hold for $n-1$ lines.

Consider first the case where $i \leqq n - p$, and therefore $[k + 2, i]$ exists for any set of $n - 1$ out of the n lines. For at least $n + k - i + 2$ of these sets of $n - 1$ lines, $[k + 2, i]$ is a key matrix (Theorem 9). Applying (A) to one of these sets of $n - 1$ lines,* it follows that there is an integer g_1, $n + k - i + 1 \leqq g_1 \leqq n - 1$, such that g_1 of the $n - 1$ lines touch a curve $P_k^{(g_1-n+k+i)}$. Similarly, for another set of $n - 1$ lines, g_2 of them will touch a curve $P_k^{(g_2-n+k+i)}$. Then at least $g_1 + g_2 - n$ lines would be tangent to both of these curves, and therefore the curves can not be distinct. For these curves, if distinct, can have at most $g_1 + g_2 + 2i - 2n - 1$ common tangents (see § 3); and from $p \geqq i$ and $i \leqq n - p$ we find $g_1 + g_2 - n > g_1 + g_2 + 2i - 2n - 1$. It follows that there is a unique curve $P_k^{(g_1-n+k+i)}$, $n + k - i + 1 \leqq g_1 \leqq n - 1$, which is touched by exactly g_1 lines out of each set of $n - 1$ lines for which $[k + 2, i]$ is a key matrix. If this is true for every set of $n - 1$ lines out of the n lines, it follows at once that $g_1 = n - 1$, and that all of the n lines touch a curve $P_k^{(k+i-1)}$. If there are sets of lines for which $[k + 2, i]$ is not a key matrix, exactly $g_1 - 1$ or $g_1 + 1$ lines† of such a set must touch the $P_k^{(g_1-n+k+i)}$. In the former case, (B) says that $[k + 2, i + 1]$ would be a key matrix for the $n - 1$ lines, and this would give $\{[k + 2, i]\} \neq 0 *$ (Theorem 4). But since $[k + 2, i]$ is a key matrix for the n lines, we have $\{[k + 2, i]\} = 0$ for all sets of $n - 1$ lines (Theorem 8). Hence if there are any sets of $n - 1$ lines for which $[k + 2, i]$ is not a key matrix, exactly $g_1 + 1$ lines of each such set must touch the $P_k^{(g_1-n+k+i)}$. In this case it is evident that exactly $g_1 + 1$ of the n lines touch this $P_k^{(g_1-n+k+i)}$, and hence (A) holds for the n lines with $g = g_1 + 1$.

We must next consider the case $i > n - p$, which only occurs when $n = 2p - 1$ and $i = p$. We wish then to show that if $[k + 2, p]$ is a key matrix for $2p - 1$ lines, $2p - 1 \geqq 3$, $0 \leqq k \leqq p - 2$; then exactly g of the lines touch a curve $P_k^{(g+k-p)}$, where $p + k + 1 \leqq g \leqq 2p - 1$.

We have

$$\begin{vmatrix} a_{k+2} & \cdots & a_p \\ \cdot & \cdot & \cdot \\ a_p & \cdots & a_{2p-k-2} \end{vmatrix} = 0,$$

and hence

$$\begin{vmatrix} c_1 t_1^{p-1} & \cdots & c_1 t_1^{k+1} & t_1^{p+k-1} & \cdots & t_1 & 1 \\ c_2 t_2^{p-1} & \cdots & c_2 t_2^{k+1} & t_2^{p+k-1} & \cdots & t_2 & 1 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ c_{2p-1} t_{2p-1}^{p-1} & \cdots & c_{2p-1} t^{k+1} & t_{2p-1}^{p+k-1} & \cdots & t_{2p-1} & 1 \end{vmatrix} = 0.$$

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* The inequality conditions of the hypothesis are readily seen to hold for the $n - 1$ lines, provided we assume, as of course we may, that $n > 3$.

† Any set of $n - 1$ lines must evidently contain either $g_1 - 1$, g_1, or $g_1 + 1$ lines touching the $P_k^{(g-n+k+i)}$; and it can not be g_1, because this, by (B), would make $[k + 2, i]$ the key matrix.

All the minors of the elements of the first column can not vanish. For they are (except for a non-vanishing factor) the determinants

$$\begin{vmatrix} \alpha_{k+3} & \cdots & \alpha_p \\ \cdot & \cdot & \cdot \\ \alpha_p & \cdots & \alpha_{2p-k-3} \end{vmatrix}$$

for the sets of $2p - 2$ out of the $2p - 1$ lines; and if they all vanished we would have, in terms of the constants for the $2p - 1$ lines,

$$\begin{vmatrix} a_{k+3} & \cdots & a_p & a_{p+1} \\ \cdot & \cdot & \cdot & \cdot \\ a_p & \cdots & a_{2p-k-3} & a_{2p-k-2} \\ t^{p-k-2} & \cdots & t & 1 \end{vmatrix} = 0$$

for $2p - 1$ different values of t. This would therefore be an identity in t, and we would have

$$\begin{vmatrix} a_{k+3} & \cdots & a_p \\ \cdot & \cdot & \cdot \\ a_p & \cdots & a_{2p-k-3} \end{vmatrix} = 0$$

for the $2p - 1$ lines, contrary to the hypothesis. Suppose the non-vanishing minor to be in the upper right-hand corner. Then the reflex point c and clinant t of each of the $2p - 1$ lines satisfies the equation

$$\text{(I)} \qquad A_1ct^{p-1} + A_2ct^{p-2} + \cdots + A_{p-k-1}ct^{k+1} + B_1t^{p+k-1} + B_2t^{p+k-2} + \cdots + B_{p+k-1}t + B_{p+k} = 0$$

where the A's and B's are the co-factors of the elements of the lower row, and $A_1 \neq 0$. Solving for ct, we have

$$ct = \frac{-B_1t^{p+k-1} - B_2t^{p+k-2} - \cdots - B_{p+k}}{A_1t^{p-2} + A_2t^{p-3} + \cdots + A_{p-k-1}t^k}$$

and multiplying numerator and denominator of the fraction by

$$\frac{(\sqrt{-1})^{p-1}}{(t_1t_2t_3\cdots t_{2p-2})^{(p+k-1)/2}},$$

they both become self-conjugate, and we have

$$\text{(II)} \qquad ct = \frac{\alpha_1t^{p+k-1} + \alpha_2t^{p+k-2} + \cdots + \bar{\alpha}_2t + \bar{\alpha}_1}{\gamma_{k+1}t^{p-2} + \gamma_{k+2}t^{p-3} + \cdots + \bar{\gamma}_{k+1}t^k}$$

where $\gamma_{k+1} \neq 0$.

The reflex point c and clinant t of each of the $2p - 1$ lines must satisfy (II), unless the clinant is such that it causes the denominator and numerator to vanish. If then the numerator and denominator have no common polynomial factor, all of the $2p - 1$ lines satisfy (II), and hence all of

them are tangents of the curve $P_k^{(k+p-1)}$ whose envelope equation is

$$tx + y = \frac{\alpha_1 t^{p+k-1} + \cdots + \bar{\alpha}_1}{\gamma_{k+1} t^{p-2} + \cdots + \bar{\gamma}_{k+1} t^k}$$

which is in accordance with part (A) of our theorem.

Suppose now that the numerator in (II) is not identically zero, and that the numerator and denominator have a highest common factor of degree $\mu + \nu$, containing t^ν (but not $t^{\nu+1}$) as a factor. It is obvious that μ and ν must satisfy the inequalities

$$0 \leqq \mu \leqq p - k - 2, \qquad 0 \leqq \nu \leqq k.$$

We may then take out this common factor and write*

$$\text{(III)} \qquad ct = \frac{\alpha_1' t^{p+k-\mu-2\nu-1} + \cdots + \bar{\alpha}_1'}{\gamma_{k-\nu-1}' t^{p-\mu-\nu-2} + \cdots + \bar{\gamma}_{k-\nu-1}' t^{k-\nu}}.$$

Each of the $2p - 1$ lines, satisfying (I), must satisfy (III) unless its clinant is such as to cause the removed common factor to vanish. Since the clinant of a line can not be zero, it follows that at least $2p - \mu - 1$ of them satisfy (III). Say that exactly $2p - \mu + \sigma - 1$ of the lines satisfy (III), $0 \leqq \sigma \leqq \mu$. These $2p - \mu + \sigma - 1$ lines evidently touch a curve $P_{k-\nu}^{(p+k-\mu-2\nu-1)}$. Then either all of the $2p - 1$ lines touch this curve, and therefore by (B) $[k - \nu + 2,\ p - \sigma - \nu]$ is a key matrix for every set of $2p - 2$ lines; or in at least one set of $2p - 2$ lines exactly $2p - \nu + \sigma - 1$ lines touch the curve, and hence $[k - \nu + 2,\ p - \sigma - \nu - 1]$ is a key for this set of $2p - 2$. In either case it follows that $[k - \nu + 2,\ p - \sigma - \nu] = 0$ for the $2p - 1$ lines (by Theorem 10 or 7). With $[k - \nu + 2,\ p - \sigma - \nu] = 0$, $\sigma \geqq 1$ would give (Theorem 1) $[k + 2,\ p - 1] = 0$, contrary to hypothesis; and hence we must have $\sigma = 0$. With $\sigma = 0$, $\nu \geqq 1$ would give $[k + 1,\ p] = 0$; and since $\nu \geqq 1$ would necessitate $k \geqq 1$, this again contradicts the hypothesis that $[k + 2,\ p]$ is a key matrix. It follows that in equation (III) we must have $\nu = 0$, and that the corresponding envelope equation

$$tx + y = \frac{\alpha_1' t^{p+k-\mu-1} + \cdots + \bar{\alpha}_1'}{\gamma_{k-1}' t^{p-\mu-2} + \cdots + \bar{\gamma}_{k-1}' t^k}$$

represents a curve $P_k^{(p+k-\mu-1)}$ which is touched by exactly $2p - \mu - 1$ of the $2p - 1$ lines. This is in agreement with part (A) of our theorem.

There remains the case where the numerator in equation (II) vanishes identically. In this case, at least $2p - 1 - (p - k - 2)$, or $p + k + 1$, of the $2p - 1$ lines are concurrent at the origin, *i.e.*, touch a $P_0^{(1)}$. Say

* The assumption that t^ν is a factor of the numerator gives $\alpha_1 = \alpha_2 = \cdots = \alpha_\nu = 0$; and when we then divide out t^ν, the degree of the numerator is reduced by 2ν.

that exactly $p + k + \sigma + 1$ of them are concurrent. As in the preceding paragraph, a contradiction will result unless $k = \sigma = 0$. If then the numerator in (II) vanishes identically, exactly $p + 1$ of the $2p - 1$ lines must touch a curve $P_0^{(1)}$.

This completes the proof that if (A) and (B) hold for $n - 1$ lines, then (A) must hold for n lines. It will now be shown that if (A) and (B) hold for $n - 1$ lines and (A) holds for n lines, then (B) also holds for n lines.

If g out of n lines touch a curve $P_k^{(g-n+k+i-1)}$, then in g of the sets of $n - 1$ out of the n lines, $g - 1$ lines touch the curve, while in the remaining $n - g$ sets, g lines touch the curve. Hence, applying (B) to the sets of $n - 1$ lines, $[k + 2, i]$ is a key matrix for g of the sets and $[k + 2, i - 1]$ for the remaining $n - g$ sets. It follows (by Theorem 7 if $g < n$, and by Theorem 10 if $g = n$) that $[k + 2, i] = 0$ for the n lines. There must then be a key matrix $[h', i']$ for the n lines, such that $i' + h' \geqq k + i + 2$ and $i' - h' \leqq i - k - 2$. Hence, applying (A) to the n lines, and proceeding as above, we have that either $[h', i']$ is a key matrix for all of the sets of $n - 1$ lines or $[h', i' - 1]$ is a key matrix for some sets of $n - 1$ lines. In the former case there would be a set of $n - 1$ lines with both $[h', i']$ and $[k + 2, i]$ as key matrices; while in the latter case there would be a set of $n - 1$ lines with key matrices $[h', i' - 1]$ and either $[k + 2, i]$ or $[k + 2, i - 1]$. Either case is a contradiction unless $[h', i']$ is $[k + 2, i]$. It follows that if g of the lines touch a curve $P_k^{(g-n+k+i-1)}$, $[k + 2, i]$ is a key matrix. If then (A) and (B) hold for $n - 1$ lines, they both hold for n lines; and this establishes our theorem.

THEOREM 12. If $[k + 2, i]$ is a key matrix for n lines, then for g of the sets of $n - 1$ out of the n lines (where $n + k - i + 2 \leqq g \leqq n$) $[k + 2, i]$ is a key matrix, and for the remaining $n - g$ sets $[k + 2, i - 1]$ is a key matrix.

This theorem is an obvious corollary of Theorem 11.

As noted at the beginning of § 3, a set of $2p$ lines is improper when, and only when, the determinant

$$\begin{vmatrix} a_3 & \cdots & a_{p+1} \\ \cdot & \cdot \quad \cdot \quad \cdot & \cdot \\ a_{p+1} & \cdots & a_{2p-1} \end{vmatrix}$$

vanishes. Since this determinant is one of the two complementary solid determinants in the matrix $[2, p]$, an improper set will occur when, and only when, $\{[2, p]\} = 0$. For $2p - 1$ lines to be improper, either or both of the conditions $[2, p] = 0$, $[3, p] = 0$, must hold. And since these vanishing matrices must lie in a vanishing area having a key matrix, it is evident that the necessary and sufficient condition for an improper set is

that there should be a key matrix $[h,\ i]$ somewhere in the area $h \geqq 2$, $i \leqq p$, $i + h \leqq p + 3$, and $i - h \geqq 0$. Since vanishing areas can not overlap, it is evident that there could not be more than one key matrix in this area. And from Theorem 11 we then have

THEOREM 13. A necessary and sufficient condition for an improper set of n lines is that there should exist a unique set of integers k, i, and g, satisfying the inequalities

$$0 \leqq k \leqq p - 2, \qquad k + i + 2 \leqq p + 3,$$

$$k + 2 \leqq i \leqq p, \qquad n + k - i + 2 \leqq g \leqq n,$$

and such that g (and no more) of the n lines are tangent to a unique curve $P_k^{(g-n+k+i-1)}$.

§ 6. *Improper Sets as Limiting Cases.*

A set of n variable lines with characteristic constants $\alpha_1, \alpha_2, \cdots, \alpha_n$ and clinants $\tau_1, \tau_2, \cdots, \tau_n$ will be said to approach a set of n fixed lines with constants $a_1, a_2, \cdots, a_n$ and clinants $t_1, t_2, \cdots, t_n$, if the α's and τ's simultaneously approach the corresponding a's and t's. Since there are only a finite number of conditions for improper sets, it is always possible (in an infinite number of ways) to find a sequence of proper sets which approach any given improper set. The question then arises as to whether, when a variable proper set thus approaches a given improper set, the Clifford figure (point or circle) of the variable set will approach, under any circumstances, a unique limit which might be conveniently defined as the Clifford figure of the improper set. This question will be answered without giving the rather tedious details of the proofs. The results may be most conveniently expressed if we assume for the time being the point of view of the geometry of inversion; *i.e.*, we assume that the plane has only one point at infinity, and understand the word circle to include straight lines and point-circles (circles of zero radius).

Type 1; $2p - 1$ lines, key matrix $[k + 2,\ p - k + 1]$, $k \geqq 1$. This key matrix lies on the skew diagonal $h + i = p + 3$; and g of the $2p - 1$ lines, $p + 2k \leqq g \leqq 2p - 1$, touch a curve $P_k^{(g-p+1)}$. In the fractions giving the center and radius of the Clifford circle, the denominators vanish while the numerators do not. Of the sets of $2p - 2$ out of the $2p - 1$ lines, g sets are proper sets; but the remaining $2p - g - 1$ sets are of the type 4 described below, with their Clifford points at infinity. In this case the g finite Clifford points lie on a straight line.* When a proper set of $2p - 1$ lines approaches such an improper set in any way, the Clifford circle approaches this straight line as a limit. We may define this line therefore as the Clifford figure in this case. The reflex point of this line is

* See Morley, loc. cit.

$$c = \begin{vmatrix} a_1 & \cdots & a_p \\ \cdot & \cdot & \cdot \\ a_p & \cdots & a_{2p-1} \end{vmatrix} \Bigg/ \begin{vmatrix} a_3 & \cdots & a_{p+1} \\ \cdot & \cdot & \cdot \\ a_{p+1} & \cdots & a_{2p-1} \end{vmatrix}.$$

The denominator of this fraction is a solid determinate of the matrix $[1, p-1]$, and can not vanish, since with $[k+2, p-k+1]$ as a key matrix we have $\{[1, p-1]\} \neq 0$. As examples of sets of this type we have 5 lines touching a deltoid; 7 lines touching a $P_2^{(4)}$; 7 lines, 6 of which touch a deltoid; etc.

Type 2; $2p-1$ lines, key matrix $[k+2, p-k]$, $k \geqq 1$. The key matrix lies on the skew diagonal $h+i = p+2$; and g of the $2p-1$ lines, $p+2k+1 \leqq g \leqq 2p-1$, touch a curve $P_k^{(g-p)}$. Both numerators and denominators of the fractions in the expression for the Clifford circle vanish. Of the sets of $2p-2$ lines, g sets are of the type 4 and the rest of the type 5 described below, with their Clifford points at infinity in both cases. When a proper set of lines approaches a set of this kind in any way, the entire Clifford circle recedes to infinity. We may therefore define the Clifford figure for this type to be a point-circle at infinity. The simplest example of a set of this type is the case of 7 lines touching a deltoid.

Type 2′; $2p-1$ lines, key matrix $[2, p]$. In this type, g of the lines, $p+1 \leqq g \leqq 2p-1$, touch a curve $P_0^{(g-p)}$; and the radius of the Clifford circle is zero.* Of the sets of $2p-2$ lines, g sets are proper, and the rest are of type 5′ with a finite Clifford point as defined below. All of these $2p-1$ Clifford points for the sets of $2p-2$ lines coincide at the point

$$\begin{vmatrix} a_1 & \cdots & a_{p-1} \\ \cdot & \cdot & \cdot \\ a_{p-1} & \cdots & a_{2p-3} \end{vmatrix} \Bigg/ \begin{vmatrix} a_3 & \cdots & a_p \\ \cdot & \cdot & \cdot \\ a_p & \cdots & a_{2p-3} \end{vmatrix},$$

which is the focus† of the curve $P_0^{(g-p)}$. When a variable proper set approaches such a set, the center of the Clifford circle approaches this point while its radius approaches zero. We may define the Clifford figure for this type to be a point-circle at this point.

Type 3; $2p-1$ lines, key matrix $[k+2, p-k-1]$, $k \geqq 1$. This key matrix lies on the skew diagonal $h+i = p+1$; and g of the lines, $p+2k+2 \leqq g \leqq 2p-1$, touch the curve $P_k^{(g-p-1)}$. Here again both the numerators and denominators of the fractions for the center and radius of the Clifford circle vanish. Of the sets of $2p-2$ lines, g sets are of type 5 with their Clifford points at infinity, and the rest are of type 6 with entirely indeterminate Clifford points. Given a set of this type, a proper set may be made to approach it in such a way that the Clifford circle will approach any arbitrarily preassigned straight line. The simplest example of this type is a set of 9 lines touching a deltoid.

* See note, beginning of § 2.

† See Clifford, loc. cit. The denominator of this fraction can not vanish.

Type 3′; $2p-1$ lines, key matrix $[2, p-1]$. In this case g of the lines, $p+2 \leqq g \leqq 2p-1$, touch a curve $P_0^{(g-p-1)}$; and again the Clifford circle fractions take the indeterminate form. Of the sets of $2p-2$ lines, g sets have a Clifford point as defined in type 5′, while the remaining sets are of type 6 with indeterminate Clifford points. The g Clifford points coincide at the point

$$\begin{vmatrix} a_1 & \cdots & a_{p-2} \\ \cdot & \cdot & \cdot \\ a_{p-2} & \cdots & a_{2p-5} \end{vmatrix} \Bigg/ \begin{vmatrix} a_3 & \cdots & a_{p-1} \\ \cdot & \cdot & \cdot \\ a_{p-1} & \cdots & a_{2p-5} \end{vmatrix},$$

which is the focus of the curve $P_0^{(g-p-1)}$. As a proper set of lines approaches a set of this type, the circumference of the Clifford circle will always approach this point. We may then say that the Clifford circle for this type is indeterminate except that it must pass through this point. For example, if a set of 7 lines touch a parabola, a proper set may be made to approach these 7 lines in such a way that the Clifford circle will approach any preassigned circle or straight line through the focus of the parabola, or a point-circle at the focus. A still simpler example is that of 5 concurrent lines.

Type 4; $2p$ lines, key matrix $[k+2, p-k+1]$, $k \geqq 1$. The key matrix is on the skew diagonal $h+i=p+3$; and g of the lines, $p+2k+1 \leqq g \leqq 2p$, touch a curve $P_k^{(g-p)}$. In the fraction representing the Clifford point the denominator vanishes while the numerator does not. Of the sets of $2p-1$ out of the $2p$ lines, g sets are of type 1 and have Clifford lines, while the remaining sets are of type 2 having Clifford point-circles at infinity. As a proper set approaches a set of this type, the Clifford point moves off to infinity; and hence one may define the point at infinity to be the Clifford point for this type. The simplest example is a set of 6 lines touching a deltoid.

The g Clifford lines for the g sets of $2p-1$ lines in this case exhibit a very interesting property which is treated in the next section.

Type 5; $2p$ lines, key matrix $[k+2, p-k]$, $k \geqq 1$. This key matrix lies in the skew diagonal $h+i=p+2$; and g of the lines, $p+2k+2 \leqq g \leqq 2p$, touch a curve $P_k^{(g-p-1)}$. The numerator and denominator of the fraction giving the Clifford point both vanish. Of the sets of $2p-1$ lines, g are of type 2 and the rest of type 3, having respectively the point at infinity and indeterminate straight lines for their Clifford circel. An iso type 4, we may define the Clifford point for this type to be the pinst at infinity. The simplest example is a set of 8 lines tangent to a deltoid.

Type 5′; $2p$ lines, key matrix $[2, p]$. Here g of the lines, $p+2 \leqq g \leqq 2p$, touch a curve $P_0^{(g-p-1)}$, and the expression for the Clifford point

takes the indeterminate form. Of the sets of $2p - 1$ lines, g sets are of type $2'$ and the rest of type $3'$; the former having Clifford point-circles and the latter indeterminate circles passing through fixed points. These points for all of the sets of $2p - 1$ lines coincide at the point

$$\begin{vmatrix} a_1 & \cdots & a_{p-1} \\ \cdot & \cdot & \cdot \\ a_{p-1} & \cdots & a_{2p-3} \end{vmatrix} \Big/ \begin{vmatrix} a_3 & \cdots & a_p \\ \cdot & \cdot & \cdot \\ a_p & \cdots & a_{2p-3} \end{vmatrix},$$

which is the focus of the curve $P_0^{(g-p-1)}$. When a proper set approaches a set of this type in any way, the Clifford point will always approach this point of coincidence; and it may therefore be defined as the Clifford point for a set of this type. As simple examples we have 4 concurrent lines; 6 lines, 5 of which are concurrent; 6 lines touching a parabola; etc.

Type 6 consists of all improper sets not included in the preceding types. The key matrix will lie somewhere above the skew diagonal $h + i = n - p + 2$. In the fractions in the expressions for the Clifford figure, both numerator and denominator will vanish. Given such an improper set of lines, one may arbitrarily preassign a Clifford figure, and then cause a proper set to approach the improper set in such a way that its Clifford figure will approach the preassigned figure. The Clifford figure for this type is therefore entirely indeterminate.

To summarize, we may say that a unique Clifford figure exists for every proper set, and, as defined in this section, for every improper set of types 1, 2, $2'$, 4, 5, and $5'$; that it is partially indeterminate for sets of type 3 and $3'$; and that it is entirely indeterminate for sets of type 6. The existence in this sense of a uniquely defined Clifford figure for a set of n lines does not imply (even for a proper set) the existence of such a determinate Clifford figure for every sub-set of lines. But it does imply

(1) That a determinate Clifford figure exists for at least $p + 1$ of the sets of $n - 1$ lines;

(2) That these Clifford figures which exist for sets of $n - 1$ lines are all incident with the Clifford figure for the n lines;

(3) That when a variable proper set of n lines approaches the given set in any way, the Clifford figure of the variable set will always approach that of the given set.

§ 7. *The Reciprocal Case.*

In type 4 of the preceding section, g of the $2p$ lines touch a curve $P_k^{(g-p)}$, and g of the sets of $2p - 1$ lines (the sets obtained by omitting one by one the g lines which touch the curve) have Clifford lines instead of circles. The reflex point of one of these lines (see type 1) is

$$c = \begin{vmatrix} a_1 & \cdots & a_p & t^p \\ \cdot & \cdot & \cdot & \cdot \\ a_p & \cdots & a_{2p-1} & t \\ a_{p+1} & \cdots & a_{2p} & 1 \end{vmatrix} \Bigg/ \begin{vmatrix} a_3 & \cdots & a_{p+1} & t^{p-1} \\ \cdot & \cdot & \cdot & \cdot \\ a_{p+1} & \cdots & a_{2p-1} & t \\ a_{p+2} & \cdots & a_{2p} & 1 \end{vmatrix}$$

where the a's are for the $2p$ lines and t is the clinant of the omitted line. With $[k + 2,\ p - k + 1]$ as key matrix, the coefficients of the highest and lowest powers of t in the denominator vanish, so that we may write

$$c = \frac{A_p t^p + A_{p-1} t^{p-1} + \cdots + A_1 t + A_0}{B_{p-k} t^{p-k} + B_{p-k-1} t^{p-k-1} + \cdots + B_{k+1} t^{k+1}}$$

where, moreover, $A_p \neq 0$ and $B_{p-k} \neq 0$. The clinant τ of this line is equal to the quotient $\bar{c}/c$. Under the condition $[k + 2,\ p - k + 1] = 0$, it may be seen that the ratio $B_s/\bar{B}_{p-s+1}$ is a constant turn for all values of s; and if we therefore put $B_s/\bar{B}_{p-s+1} = T$, we find

$$\tau = \frac{\bar{c}}{c} = (-S)^p T t,$$

where S is the product of the $2p$ clinants of the lines of the set. We may then put $t = \tau/(-S)^p T$, and thus obtain an expression for the reflex point c of the line in terms of its clinant τ, thus

$$c = \frac{C_p t^p + C_{p-1} t^{p-1} + \cdots + C_1 t + C_0}{D_{p-k} t^{p-k} + \cdots + D_{k+1} t^{k+1}},$$

where

$$C_j = A_j/(-S)^{pj} T^j \qquad \text{and} \qquad D_s = B_s/(-S)^{ps} T^s.$$

If we now multiply both sides of the equation by τ, and the numerator and denominator of the fraction by $(-S)^{(p^2+p)/2} T^{p/2}$, we obtain

$$c\tau = \frac{\alpha_1 \tau^p + \alpha_2 \tau^{p-1} + \cdots + \bar{\alpha}_2 \tau - \bar{\alpha}_1}{\gamma_{k+1} \tau^{p-k-1} + \cdots + \bar{\gamma}_{k+1} \tau^k},$$

where the numerator and denominator are now found to be self conjugate, and $\alpha_1 \neq 0$ and $\gamma_{k+1} \neq 0$. Hence we see that these g Clifford lines are tangents of the curve $P_k^{(p)}$ whose envelope equation is

$$tx + y = \frac{\alpha_1 \tau^p + \alpha_2 \tau^{p-1} + \cdots + \bar{\alpha}_1}{\gamma_{k+1} \tau^{p-k-1} + \cdots + \bar{\gamma}_{k+1} \tau^k}.$$

The most interesting case occurs when $g = 2p$, *i.e.*, when all of the $2p$ lines touch a curve $P_k^{(p)}$. We then obtain $2p$ Clifford lines for the sets of $2p - 1$ lines, and these Clifford lines touch a new curve $P_k^{(p)}$. An examination of the incidence relations of the various Clifford points and circles for the subsets of lines shows that the relation between the two

sets of $2p$ lines is reciprocal; *i.e.*, if we start with the $2p$ Clifford lines, we are led back to the original set.*

There is also a certain reciprocity in the cases where $g < 2p$. For instance, when $g = 2p - 1$, $2p - 1$ of the $2p$ lines touch a $P_k^{(p-1)}$, and we are led to $2p - 1$ Clifford lines touching a $P_k^{(p)}$. Reversing the process, and starting with these $2p - 1$ lines, we have a case of type 1 with a Clifford line, this line being the one line of the original set which does not touch the $P_k^{(p-1)}$. For $g < 2p - 1$, the fact that the g Clifford lines touch a $P_k^{(p)}$ is not sufficient to make them an improper set;† and hence they will, in general, be a proper set and have a Clifford element which will be the Clifford element of the $2p - g$ lines of the original set which do not touch the $P_k^{(g-p)}$.

For example, suppose that 19 of 24 lines touch a curve $P_3^{(7)}$, while the remaining 5 lines have no exceptional conditions upon them. Then 19 of the sets of 23 lines will be of type 1 and have Clifford lines, and these 19 Clifford lines will touch a $P_3^{(12)}$. Starting with these 19 lines, they will be found to be a proper set, their Clifford circle being that of the 5 lines of the original set which did not touch the $P_3^{(7)}$. But if the 5 lines of the original set touch a parabola, and hence have a point-circle as their Clifford figure, then the 19 lines in the reverse process will lead to this same point-circle as their Clifford figure, and will therefore be of type $2'$; and it follows that in addition to the fact that the 19 Clifford lines touch a curve $P_3^{(12)}$, g of them, where $11 \gtreqless g \gtreqless 19$, must also touch a curve $P_0^{(g-10)}$.

§ 8. *The Curve $P_k^{(m)}$ as a Sum of Fundamental Curves.*

Let two curves $P_{k_1}^{(m_1)}$ and $P_{k_2}^{(m_2)}$ be given by the envelope equations

$$tx + y = \frac{\alpha_1(t)}{\gamma_1(t)} \qquad \text{and} \qquad tx + y = \frac{\alpha_2(t)}{\gamma_2(t)}\,;$$

and suppose that $\gamma_1(t)$ and $\gamma_2(t)$ have no common polynomial factor, which necessitates that one of the k's, say k_2, should be zero. Then the envelope equation

$$tx + y = \frac{\alpha_1(t)}{\gamma_1(t)} + \frac{\alpha_2(t)}{\gamma_2(t)} = \frac{\alpha_1(t)\gamma_2(t) + \alpha_2(t)\gamma_1(t)}{\gamma_1(t)\gamma_2(t)}$$

represents a curve $P_{k_1}^{(m_1+m_2-1)}$ which may be called the "sum" of the two curves. Since the map equations of the three curves $P_{k_1}^{(m_1)}$, $P_0^{(m_2)}$, and $P_{k_1}^{(m_1+m_2-1)}$ are respectively

* Kantor (loc. cit.) discovered this property for a set of 6 lines touching a deltoid. Morley generalized Kantor's theorem; but his generalization is quite different from the one here given.

† For $g \gtreqless 2p - 2k$, the conclusion that the g lines touch a curve $P_k^{(p)}$ is obviously trivial.

$$x = \frac{d}{dt}\left(\frac{\alpha_1(t)}{\gamma_1(t)}\right), \quad x = \frac{d}{dt}\left(\frac{\alpha_2(t)}{\gamma_2(t)}\right), \quad \text{and} \quad x = \frac{d}{dt}\left(\frac{\alpha_1(t)}{\gamma_1(t)}\right) + \frac{d}{dt}\left(\frac{\alpha_2(t)}{\gamma_2(t)}\right),$$

and since in all three cases the parameter t is the clinant of the tangent to the curve at the point x, the geometric significance of the addition is obvious. If we take two points, one on the curve $P_{k_1}^{(m_1)}$ and the other on the curve $P_0^{(m_2)}$, where the tangents to the curves are parallel, and add these points by the ordinary construction for the sum of two complex numbers; the result will be the point of the curve $P_{k_1}^{(m_1+m_2-1)}$ at which the tangent has the same direction.* We may similarly define the sum of any number of such curves, all the denominators in the envelope equations being prime to each other.

We may call a curve $P_k^{(m)}$ a "fundamental curve" when the denominator $\gamma(t)$ in the envelope equation is a power of a self-conjugate factor of the first degree or a power of a self-conjugate factor of the second degree which is not the product of self-conjugate factors of the first degree. For such a fundamental curve we have $k = 0$, with $\gamma(t)$ of the form $(\gamma t + \bar{\gamma})^{m-1}$ or $(\gamma t^2 + rt + \bar{\gamma})^{(m-1)/2}$; or† $k = \frac{1}{2}(m - 1)$, with $\gamma(t)$ of the form rt^k. The map equations in the three cases may be written respectively in the forms

$$x = A_0 + \frac{B_2}{(\gamma t + \bar{\gamma})^2} + \frac{B_3}{(\gamma t + \bar{\gamma})^3} + \cdots + \frac{B_m}{(\gamma t + \bar{\gamma})^m},$$

$$x = A_0 + \frac{B_{m-1}t^{m-1} + B_{m-2}t^{m-2} + \cdots + B_1 t + B_0}{(\gamma t^2 + rt + \bar{\gamma})^{m+1/2}},$$

$$x = A_k t^k + A_{k-1}t^{k-1} + \cdots + A_1 t + A_0 + \frac{B_1}{t} + \frac{B_2}{t^2} + \cdots + \frac{B_{k+1}}{t^{k+1}}.$$

In the first two cases the point A_0 is the focus of the curve; while in the third case it may be called the center (suggested by the center of the deltoid when $k = 1$). When $A_0 = 0$, *i.e.*, when the focus or center is at the origin, the curve may be said to be in canonical position. Every curve $P_k^{(m)}$ is the sum of q such fundamental curves,‡ where q is the number of distinct irreducible self-conjugate factors of $\gamma(t)$. It is evident that if we had such a decomposition of a curve $P_k^{(m)}$ into a sum of fundamental curves, the A_0's for the several fundamental curves would be arbitrary except that their sum would be fixed. The point represented by the sum of the A_0's may be called (for lack of a better name) the center of the $P_k^{(m)}$; and the $P_k^{(m)}$ may be said to be in canonical position when its center is at the origin. If a curve $P_k^{(m)}$ is in canonical position, its decomposition into the sum of fundamental curves in canonical position is unique.

* These curves have one and only one tangent in a given direction.

† This case $k = \frac{1}{2}(m - 1)$ is the curve $\Delta^{(2n-1)}$ of Morley and Stephens (loc. cit.).

‡ Theorems 5 and 6 of Atchison's paper (loc. cit.) are special cases of this.